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On Frobenius incidence varieties of linear subspaces over finite fields

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ABSTRACT

We define Frobenius incidence varieties by means of the incidence relation of Frobenius images of linear subspaces in a fixed vector space over a finite field, and investigate their properties such as supersingularity, Betti numbers and unirationality. These varieties are variants of the Deligne–Lusztig varieties. We then study the lattices associated with algebraic cycles on them. We obtain a positive-definite lattice of rank 84 that yields a dense sphere packing from a 4-dimensional Frobenius incidence variety in characteristic 2.

1. Introduction

Codes arising from the rational points of Deligne–Lusztig varieties have been studied in several cases [14,15,20]. In this paper, we investigate lattices arising from algebraic cycles on certain variants of Deligne–Lusztig varieties, which we call *Frobenius incidence varieties*. We study basic properties of Frobenius incidence varieties such as supersingularity, Betti numbers and unirationality. By means of intersection pairing of algebraic cycles on a 4-dimensional Frobenius incidence variety over $\mathbb{F}_2$, we obtain a positive-definite lattice of rank 84 that yields a dense sphere packing.

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1.1. An illustrating example

Before giving the general definition of Frobenius incidence varieties in Section 1.2, we present the simplest example of Frobenius incidence *surfaces*, hoping that it explains the motivation for the main results of this paper.

We fix a vector space V over $\mathbb{F}_p$ of dimension 3 with coordinates (x_1, x_2, x_3), and consider the projective plane $\mathbb{P}_*(V)$ with the homogeneous coordinate system $(x_1 : x_2 : x_3)$. Let $\bar{F}$ be an algebraic closure of $\mathbb{F}_p$. An $\bar{F}$-valued point $(a_1 : a_2 : a_3)$ of $\mathbb{P}_*(V)$ corresponds to the 1-dimensional linear subspace of $V \otimes \bar{F}$ spanned by $(a_1, a_2, a_3) \in V \otimes \bar{F}$. Let $\mathbb{P}^*(V)$ denote the dual projective plane with homogeneous coordinates $(y_1 : y_2 : y_3)$ dual to $(x_1 : x_2 : x_3)$. An $\bar{F}$-valued point $(b_1 : b_2 : b_3)$ of $\mathbb{P}^*(V)$ corresponds to the 2-dimensional linear subspace of $V \otimes \bar{F}$ defined by $b_1x_1 + b_2x_2 + b_3x_3 = 0$. The incidence variety is a hypersurface of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$ defined by $x_1y_1 + x_2y_2 + x_3y_3 = 0$, which parametrizes all the pairs (L, M) of a 1-dimensional linear subspace L and a 2-dimensional linear subspace M such that $L \subset M$.

Let q be a power of p by a positive integer. The qth power Frobenius morphism of $V \otimes \bar{F}$ is the morphism from $V \otimes \bar{F}$ to itself given by $(x_1, x_2, x_3) \mapsto (x_1^q, x_2^q, x_3^q)$. For a linear subspace N of $V \otimes \bar{F}$, we denote by $N^q \subset V \otimes \bar{F}$ the image of N by the qth power Frobenius morphism, which is again a linear subspace of $V \otimes \bar{F}$. If a 2-dimensional linear subspace M of $V \otimes \bar{F}$ corresponds to a point $(b_1 : b_2 : b_3) \in \mathbb{P}^*(V)$, the linear subspace M^q corresponds to the point $(b_1^q : b_2^q : b_3^q)$.

We take two Frobenius twists of the incidence variety, and take their intersection. Let r and s be powers of p by positive integers. The hypersurface of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$ defined by

$$x_1^r y_1 + x_2^r y_2 + x_3^r y_3 = 0 \tag{1.1}$$

parametrizes the pairs (L, M) such that $L^r \subset M$, while the hypersurface of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$ defined by

$$x_1 y_1^s + x_2 y_2^s + x_3 y_3^s = 0 \tag{1.2}$$

parametrizes the pairs (L, M) such that $L \subset M^s$. Using affine coordinates of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$, we see that these two hypersurfaces (1.1) and (1.2) intersect transversely. Let X be the intersection, which is a smooth surface parameterizing the pairs (L, M) such that

$$L^r \subset M \quad \text{and} \quad L \subset M^s,$$

or equivalently

$$L^r \subset M \cap M^{rs},$$

or equivalently

$$L + L^{rs} \subset M^s.$$

We put $q := rs$, and count the $\mathbb{F}_{q^\nu}$-rational points of the surface X for positive integers ν, that is, we count the number of the pairs (L, M) of $\mathbb{F}_{q^\nu}$-rational linear subspaces L and M that satisfy the above conditions. Consider the first projection $\pi_1 : X \to \mathbb{P}_*(V)$. Let P be an $\mathbb{F}_{q^\nu}$-rational point of $\mathbb{P}_*(V)$ corresponding to $L \subset V \otimes \bar{F}$. Then, if $\dim(L + L^q) = 2$, the fiber $\pi_1^{-1}(P)$ consists of a single $\mathbb{F}_{q^\nu}$-rational point corresponding to the $\mathbb{F}_{q^\nu}$-rational subspace M such that $L + L^q = M^s$, while, if $\dim(L + L^q) = 1$, it is isomorphic to an $\mathbb{F}_{q^\nu}$-rational projective line parameterizing subspaces M such that $L + L^q \subset M^s$. Since $\dim(L + L^q) = 1$ holds if and only P is an $\mathbb{F}_q$-rational point of $\mathbb{P}_*(V)$, the number of the $\mathbb{F}_{q^\nu}$-rational points of X is equal to

$$\left(\frac{q^{3\nu} - 1}{q^\nu - 1} - \frac{q^3 - 1}{q - 1}\right) + \left(\frac{q^{2\nu} - 1}{q^\nu - 1}\right) \cdot \left(\frac{q^3 - 1}{q - 1}\right).$$

If we put

$$N(t) := t^2 + \left(q^2 + q + 2\right)t + 1,$$

then this number is equal to $N(q^\nu)$. In particular, from the classical theorems on the Weil conjecture (see, for example, [12, App. C]), we obtain the Betti numbers $b_i(X)$ of the surface X. We have $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$ and

$$b_2(X) = q^2 + q + 2.$$

Remark that, when $r > 2$ and $s > 2$, the canonical line bundle $\mathcal{O}(r-2, s-2)$ of X is ample and has non-zero global sections. Hence, the complex algebraic surface $X_{\mathbb{C}}$ defined by (1.1) and (1.2) in $\mathbb{CP}^2 \times \mathbb{CP}^2$ cannot be unirational (see [12, Chap. V, Remark 6.2.1]), and the second Betti cohomology group of $X_{\mathbb{C}}$ cannot be spanned by the classes of algebraic cycles because of the Hodge-theoretic reason (see [11, p. 163]).

However, the surface X has the so-called pathological properties of algebraic varieties in positive characteristics, that is, X contradicts naive expectations from the properties of $X_{\mathbb{C}}$. Since the projection $\pi_1 : X \to \mathbb{P}_*(V)$ gives rise to a purely inseparable extension of the function fields, X is unirational. Moreover, since $N(t)$ is a polynomial in t, the eigenvalues of the qth power Frobenius endomorphism acting on the l-adic cohomology ring of X is a power of q by integers. According to the Tate conjecture, the second l-adic cohomology group of X should be spanned by the classes of algebraic curves on X defined over $\mathbb{F}_q$. This is indeed the case. There are $2(b_2(X) - 1)$ special rational curves defined over $\mathbb{F}_q$ on X; the fibers Σ_P of $\pi_1 : X \to \mathbb{P}_*(V)$ over the $\mathbb{F}_q$-rational points P of $\mathbb{P}_*(V)$, and the fibers Σ'_Q of $\pi_2 : X \to \mathbb{P}^*(V)$ over the $\mathbb{F}_q$-rational points Q of $\mathbb{P}^*(V)$. By calculating the intersection numbers between these curves (see [12, Chap. V, §1]), we see that the numerical equivalence classes of Σ_P and Σ'_Q together with the classes of the line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ form a hyperbolic lattice $\mathcal{N}(X)$ of rank $b_2(X)$ under the intersection pairing. Thus their classes span the second l-adic cohomology group of X.

When $p = r = s = 2$, the surface X is a supersingular $K3$ surface in characteristic 2 with $|\operatorname{disc} \mathcal{N}(X)| = 4$. The defining equations (1.1) and (1.2), which were discovered by Mukai, and the configuration of the $21 + 21$ rational curves Σ_P and Σ'_Q played an important role in the study of the automorphism group of this $K3$ surface in Dolgachev and Kondō [6].

Looking at this example, we expect that the lattice $\mathcal{N}(X)$ possesses interesting properties. In particular, its primitive part can yield a dense sphere packing.

1.2. Definitions and the main results

We give the definition of Frobenius incidence varieties, and state the main results of this paper.

Let p be a prime, and let $q := p^\nu$ be a power of p by a positive integer ν. For a field F of characteristic p with an algebraic closure $\bar{F}$, we put

$$F^q := \left\{x^q \;\middle|\; x \in F\right\} \quad \text{and} \quad F^{1/q} := \left\{x \in \bar{F} \;\middle|\; x^q \in F\right\}.$$

For a scheme Y defined over a subfield of F, we denote by $Y(F)$ the set of F-valued points of Y.

We fix an n-dimensional linear space V over $\mathbb{F}_p$ with $n \geqslant 3$, and denote by $G_{n,l} = G_n^{n-l}$ or by $G_{V,l} = G_V^{n-l}$ the Grassmannian variety of l-dimensional subspaces of V. To ease the notation, we use the same letter L to denote an F-valued point $L \in G_{n,l}(F)$ of $G_{n,l}$ and the corresponding linear subspace $L \subset V_F := V \otimes F$. Moreover, for an extension field F' of F, we write L for the linear subspace $L \otimes_F F'$ of $V_{F'}$. Let ϕ denote the pth power Frobenius morphism of $G_{n,l} \otimes \bar{\mathbb{F}}_p$ over $\bar{\mathbb{F}}_p$, and let $\phi^{(q)}$ be the ν-fold iteration of ϕ. Then $\phi^{(q)}$ induces a bijection from $G_{n,l}(F)$ to $G_{n,l}(F^q)$. We denote by $L^q \in G_{n,l}(F^q)$ the image of $L \in G_{n,l}(F)$ by $\phi^{(q)}$, and by $L^{1/q} \in G_{n,l}(F^{1/q})$ the point that is mapped to L by $\phi^{(q)}$. Let $(x_1, \dots, x_n)$ be $\mathbb{F}_p$-rational coordinates of V. If L is defined in V_F by linear equations

$\sum_j a_{ij}x_j = 0$ $(i = 1, \dots, n-l)$ with $a_{ij} \in F$, then L^q is defined by the linear equations $\sum_j a_{ij}^q x_j = 0$ $(i = 1, \dots, n-l)$.

Let l and c be positive integers such that $l + c < n$. We denote by $\mathcal{I}$ the incidence subvariety of $G_{n,l} \times G_n^c$. By definition, $\mathcal{I}$ is the reduced subscheme of $G_{n,l} \times G_n^c$ such that, for any field F of characteristic p, we have

$$\mathcal{I}(F) = \{(L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L \subset M\}.$$

Let r and s be powers of p by non-negative integers such that $r > 1$ or $s > 1$ holds. We define the *Frobenius incidence variety* $X[r,s]_l^c$ to be the scheme-theoretic intersection of the pull-backs $(\phi^{(r)} \times \mathrm{id})^* \mathcal{I}$ and $(\mathrm{id} \times \phi^{(s)})^* \mathcal{I}$ of $\mathcal{I}$, where id and $\phi^{(1)}$ denote the identity map:

$$X[r,s]_l^c := \left(\phi^{(r)} \times \mathrm{id}\right)^* \mathcal{I} \cap \left(\mathrm{id} \times \phi^{(s)}\right)^* \mathcal{I}.$$

For simplicity, we write X or $X[r,s]$ or X_l^c for $X[r,s]_l^c$ if there is no possibility of confusion. The scheme X is defined over $\mathbb{F}_p$ and, for any field F over $\mathbb{F}_p$, we have

$$X(F) = \{(L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L^r \subset M \text{ and } L \subset M^s\}. \tag{1.3}$$

We have the following:

Proposition 1.1. *The projective scheme X is smooth and geometrically irreducible of dimension $(n-l-c)(l+c)$.*

Example 1.2. Let $(x_1 : \dots : x_n)$ and $(y_1 : \dots : y_n)$ be homogeneous coordinates of $G_{V,1} = \mathbb{P}_*(V)$ and $G_V^1 = \mathbb{P}^*(V)$, respectively, that are dual to each other. Then the incidence subvariety $\mathcal{I}$ is defined by $\sum x_i y_i = 0$ in $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$, and hence $X[r,s]_1^1$ is defined by

$$\begin{cases} x_1^r y_1 + \dots + x_n^r y_n = 0, \\ x_1 y_1^s + \dots + x_n y_n^s = 0. \end{cases} \tag{1.4}$$

Therefore $X[r,s]_1^1$ is of general type when r and s are sufficiently large.

We show that the Frobenius incidence varieties, which are of non-negative Kodaira dimension in general, have two pathological features of algebraic geometry in positive characteristics; namely, supersingularity and unirationality.

Our first main result is as follows:

Theorem 1.3. *There exists a polynomial $N(t)$ with integer coefficients such that the number of $\mathbb{F}_{(rs)^\nu}$-rational points of X is equal to $N((rs)^\nu)$ for any $\nu \in \mathbb{Z}_{>0}$.*

In other words, X is supersingular over $\mathbb{F}_{rs}$ in the sense that the eigenvalues of the rsth power Frobenius endomorphism acting on the l-adic cohomology ring of $X \otimes \bar{\mathbb{F}}_{rs}$ are powers of rs by integers.

We also give in Theorem 2.2 a recursive formula for the polynomial $N(t)$. We see that the odd Betti numbers of X are zero, and can calculate the even Betti numbers $b_{2i}(X)$ of X via the formula

$$N(t) = \sum_{i=0}^{\dim X} b_{2i}(X) t^i. \tag{1.5}$$

Example 1.4. The Betti numbers of $X[r,s]_1^1$ in Example 1.2 are

$$b_{2i} = b_{2(n-2)-2i} = \begin{cases} i+1 & \text{if } i < n-2, \\ n-2+((rs)^n-1)/(rs-1) & \text{if } i = n-2. \end{cases}$$

The number of rational points of the Deligne–Lusztig varieties has been studied by means of the representation theory of algebraic groups over finite fields. See, for example, [27] and [2]. Our proof of Theorems 1.3 and 2.2 does not use the representation theory, and is entirely elementary.

Our next result is on the unirationality of the Frobenius incidence varieties. A variety Y defined over $\mathbb{F}_q$ is said to be *purely-inseparably unirational over* $\mathbb{F}_q$ if there exists a purely inseparable dominant rational map $\mathbb{P}^N \cdots\to Y$ defined over $\mathbb{F}_q$.

Theorem 1.5. *The Frobenius incidence variety X is purely-inseparably unirational over $\mathbb{F}_p$.*

The relation of supersingularity to unirationality has been observed in various cases. See Shioda [24] for the supersingularity of unirational surfaces, and see Shioda and Katsura [26] and Shimada [22] for the unirationality of supersingular Fermat varieties.

From the defining equations (1.4) of $X[r,s]_1^1$, we see that $X[r,s]_1^1$ is a complete intersection of two varieties of unseparated flags [13], or more specifically, of two unseparated incidence varieties [17, §2]. Varieties of unseparated flags are classified in [28] and [13]. Their pathological property with respect to Kodaira vanishing theorem was studied in [16].

Next we investigate algebraic cycles on the Frobenius incidence varieties. Let Λ be an $\mathbb{F}_{rs}$-rational linear subspace of $V_{\mathbb{F}} := V \otimes \mathbb{F}_{rs}$ such that $l \leqslant \dim \Lambda \leqslant n-c$. We define a subvariety Σ_Λ of $G_{n,l} \times G_n^c$ by

$$\Sigma_\Lambda := G_{\Lambda,l} \times G^c_{V_{\mathbb{F}}/\Lambda^{(r)}}. \tag{1.6}$$

Then Σ_Λ is defined over $\mathbb{F}_{rs}$ and, for any field F over $\mathbb{F}_{rs}$, we have

$$\Sigma_\Lambda(F) = \{(L,M) \in G_{n,l}(F) \times G_n^c(F) \mid L \subset \Lambda \text{ and } \Lambda^r \subset M\}.$$

It follows from $\Lambda^{rs} = \Lambda$ that Σ_Λ is contained in X. In Theorem 4.1, we calculate the intersection of these algebraic cycles Σ_Λ in the Chow ring $A(X)$ of X. (See [12, App. A] or [10] for the definition of Chow rings.)

Applying Theorem 4.1 to the case $l = c = 1$, we investigate the lattice generated by the numerical equivalence classes of middle-dimensional algebraic cycles of

$$X_1^1 = X[r,s]_1^1 \subset G_{V,1} \times G_V^1 = \mathbb{P}_*(V) \times \mathbb{P}^*(V).$$

Note that, when $l = c$, we have $2\dim \Sigma_\Lambda = \dim X[r,s]_l^l$ for any Λ. Let $A^{n-2}(X_1^1)$ denote the Chow group of middle-dimensional algebraic cycles on X_1^1 over $\bar{\mathbb{F}}_p$. For $i = 1, \dots, n-1$, let h_i be the intersection of X_1^1 with $P_i \times P_{n-i} \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V)$, where P_j is a general j-dimensional projective linear subspace of $\mathbb{P}_*(V)$ or $\mathbb{P}^*(V)$. Then h_i is of middle-dimension on X_1^1. We consider the submodule

$$\widetilde{\mathcal{N}}(X_1^1) \subset A^{n-2}(X_1^1)$$

generated by $h_1, \dots, h_{n-1}$ and Σ_Λ associated with all $\mathbb{F}_{rs}$-rational linear subspaces Λ of $V \otimes \mathbb{F}_{rs}$ such that $1 \leqslant \dim \Lambda \leqslant n-1$. Then we have the intersection pairing on $\widetilde{\mathcal{N}}(X_1^1)$. Let $\widetilde{\mathcal{N}}(X_1^1)^\perp$ denote the submodule of $\widetilde{\mathcal{N}}(X_1^1)$ consisting of $x \in \widetilde{\mathcal{N}}(X_1^1)$ such that $(x,y) = 0$ holds for any $y \in \widetilde{\mathcal{N}}(X_1^1)$. We set

$$\mathcal{N}(X_1^1) := \widetilde{\mathcal{N}}(X_1^1)/\widetilde{\mathcal{N}}(X_1^1)^\perp.$$

Then $\mathcal{N}(X_1^1)$ is a finitely-generated free $\mathbb{Z}$-module equipped with the non-degenerate intersection pairing

$$\mathcal{N}(X_1^1) \times \mathcal{N}(X_1^1) \to \mathbb{Z}.$$

Thus $\mathcal{N}(X_1^1)$ is a lattice. In Theorem 5.1, we describe the rank and the discriminant of this lattice. As a corollary of Theorem 5.1, we obtain the following:

Corollary 1.6. *The l-adic cohomology ring of $X_1^1 \otimes \bar{\mathbb{F}}_{rs}$ is generated by the cohomology classes of the algebraic cycles Σ_Λ and the image of the restriction homomorphism from the cohomology ring of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$.*

In Theorem 5.1, it is shown that the discriminant of $\mathcal{N}(X_1^1)$ is a power of p. This fact is an analogue of the theorem on the discriminant of the Néron–Severi lattice of a supersingular $K3$ surface (in the sense of Shioda) proved by Artin [1] and Rudakov and Shafarevich [21]. See also [23] for a similar result on the Fermat variety of degree $q+1$.

For $x \in \widetilde{\mathcal{N}}(X_1^1)$, let $[x] \in \mathcal{N}(X_1^1)$ denote the class of x modulo $\widetilde{\mathcal{N}}(X_1^1)^\perp$. We define the primitive part $\mathcal{N}_{\mathrm{prim}}(X_1^1)$ of $\mathcal{N}(X_1^1)$ by

$$\mathcal{N}_{\mathrm{prim}}(X_1^1) := \bigl\{[x] \in \mathcal{N}(X_1^1) \bigm| \bigl([x], [h_i]\bigr) = 0 \text{ for } i = 1, \dots, n-1\bigr\}.$$

For a lattice L, let $[-1]^\nu L$ denote the lattice obtained from L by multiplying the symmetric bilinear form with $(-1)^\nu$.

Theorem 1.7. *The intersection pairing on $\mathcal{N}_{\mathrm{prim}}(X_1^1)$ is non-degenerate. The lattice $[-1]^n \mathcal{N}_{\mathrm{prim}}(X_1^1)$ is positive-definite of rank $|\mathbb{P}^{n-1}(\mathbb{F}_{rs})| - 1$.*

In the last section, our construction is applied to the sphere packing problem. We investigate the case $n = 4$, and study the positive-definite lattice $\mathcal{N}_{\mathrm{prim}}(X[2,2]_1^1)$ of the 4-dimensional Frobenius incidence variety $X[2,2]_1^1$.

Theorem 1.8. *Suppose that $n = 4$. The lattice $\mathcal{N}_{\mathrm{prim}}(X[2,2]_1^1)$ is an even positive-definite lattice of rank* 84*, with discriminant* $85 \cdot 2^{16}$*, and with minimal norm* 8*.*

In particular, the normalized center density of $\mathcal{N}_{\mathrm{prim}}(X[2,2]_1^1)$ is $2^{34}/\sqrt{85} = 2^{30.795\ldots}$, while the Minkowski–Hlawka bound at rank 84 is $2^{17.546\ldots}$. See Section 6 for the definition of normalized center density and Minkowski–Hlawka bound.

In the proof of Theorem 1.8, we construct another positive-definite lattice $\mathcal{M}_{\mathcal{C}}$ of rank 85 associated with a code $\mathcal{C}$ over $\mathbb{Z}/8\mathbb{Z}$. The normalized center density $2^{32.5}$ of $\mathcal{M}_{\mathcal{C}}$ is also larger than the Minkowski–Hlawka bound $2^{18.429\ldots}$ at rank 85. See Theorem 6.1.

1.3. The plan of this paper

The proofs of these results are given as follows. In Section 2, we show that the Frobenius incidence variety X is smooth in Proposition 2.1, and give a recursive formula for the number of $\mathbb{F}_{(rs)^\nu}$-rational points of X in Theorem 2.2. Proposition 1.1 and Theorem 1.3 follow from these results. In Section 3, we show that X is purely-inseparably unirational. In Section 4, we give a formula for the intersection of the algebraic cycles Σ_Λ in the Chow ring of X. In Section 5, we study the case where $l = c = 1$, and prove Corollary 1.6 and Theorem 1.7. In the last section, we study the case $n = 4$, $l = c = 1$, $r = s = 2$, and prove Theorem 1.8.

2. Number of rational points and the Betti numbers

In this section, we prove Proposition 1.1 and Theorem 1.3.

It is useful to note that the defining property (1.3) of the Frobenius incidence variety $X = X[r,s]^c_l$ is rephrased as follows:

$$\begin{aligned} X(F) &= \{(L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L^r \subset M \text{ and } L \subset M^s\} \\ &= \{(L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L + L^{rs} \subset M^s\} \\ &= \{(L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L^r \subset M \cap M^{rs}\}. \end{aligned} \tag{2.1}$$

We denote by

$$S_{n,l} \to G_{n,l} \quad \text{and} \quad Q^c_n \to G^c_n$$

the universal subbundle of $V \otimes \mathcal{O} \to G_{n,l}$ and the universal quotient bundle of $V \otimes \mathcal{O} \to G^c_n$, respectively. We consider the vector bundle

$$\mathcal{E} := \mathcal{H}om\big(\mathrm{pr}^*(S_{n,l}), \mathrm{pr}^*\big(Q^c_n\big)\big) \to G_{n,l} \times G^c_n$$

of rank lc, where pr denotes the projections $G_{n,l} \times G^c_n \to G_{n,l}$ and $G_{n,l} \times G^c_n \to G^c_n$. Let $\gamma : G_{n,l} \times G^c_n \to \mathcal{E}$ denote the section of $\mathcal{E}$ corresponding to the canonical homomorphism

$$\mathrm{pr}^*(S_{n,l}) \hookrightarrow V \otimes \mathcal{O}_{G_{n,l} \times G^c_n} \twoheadrightarrow \mathrm{pr}^*\big(Q^c_n\big).$$

We then put

$$\mathcal{F} := \big(\phi^{(r)} \times \mathrm{id}\big)^* \mathcal{E} \oplus \big(\mathrm{id} \times \phi^{(s)}\big)^* \mathcal{E}, \tag{2.2}$$

which is a vector bundle over $G_{n,l} \times G^c_n$ of rank $2lc$ that has a canonical section

$$\tilde{\gamma} := \big(\big(\phi^{(r)} \times \mathrm{id}\big)^* \gamma, \big(\mathrm{id} \times \phi^{(s)}\big)^* \gamma\big) : G_{n,l} \times G^c_n \to \mathcal{F}.$$

Since the incidence variety $\mathcal{I}$ is defined on $G_{n,l} \times G^c_n$ by $\gamma = 0$, the subscheme X of $G_{n,l} \times G^c_n$ is defined by $\tilde{\gamma} = 0$.

Proposition 2.1. *The section $\tilde{\gamma}$ intersects the zero section of $\mathcal{F}$ transversely in the total space of $\mathcal{F}$. In particular, the scheme X is smooth of dimension*

$$\dim\big(G_{n,l} \times G^c_n\big) - 2lc = (l + c)(n - l - c).$$

Proof. It is enough to show that, for any field F of characteristic p, the tangent space to X at an arbitrary F-valued point of X is of dimension $(l + c)(n - l - c)$.

Let (L, M) be an F-valued point of $\mathcal{I}$. Then the tangent space to $G_{n,l} \times G^c_n$ at (L, M) is canonically identified with the linear space

$$T(L, M) := \mathrm{Hom}(L, V_F/L) \oplus \mathrm{Hom}(M, V_F/M),$$

and the tangent space to $\mathcal{I}$ at (L, M) is identified with the linear subspace of $T(L, M)$ consisting of pairs $(\alpha, \beta) \in T(L, M)$ that make the following diagram commutative:

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\ V_F/L \arrow[r, two heads] & V_F/M, \end{tikzcd} \tag{2.3}$$

where the horizontal arrows are the natural linear maps.

We now let (L, M) be an F-valued point of X. Note that the Frobenius morphism induces the zero map on the tangent space. Suppose that $r > 1$ and $s > 1$. Then the tangent space to X at (L, M) is identified with the linear subspace of $T(L, M)$ consisting of pairs (α, β) that make both triangles

$$(\mathrm{T}_\beta) \quad \begin{tikzcd} L \arrow[r, hook] \arrow[dr, "0"'] & M \arrow[d, "\beta"] \\ & V_F/M \end{tikzcd} \quad \text{and} \quad (\mathrm{T}_\alpha) \quad \begin{tikzcd} L \arrow[d, "\alpha"'] \arrow[dr, "0"] & \\ V_F/L \arrow[r, two heads] & V_F/M^s \end{tikzcd}$$

commutative. Suppose that $r = 1$ and $s > 1$ (resp. $r > 1$ and $s = 1$). Then the tangent space to X at (L, M) is identified with the linear subspace of pairs (α, β) that make both of (2.3) and (T_α) (resp. (2.3) and (T_β)) commutative. In each case, one easily checks that the dimension of the tangent space is $(l + c)(n - l - c)$. $\square$

Next we count the number of $\mathbb{F}_{(rs)^\nu}$-rational points of X. In order to state the result, we need to introduce several polynomials.

For each integer l with $0 \leqslant l \leqslant n$, we define a polynomial $g_{n,l}(x) = g_n^{n-l}(x) \in \mathbb{Z}[x]$ of degree $l(n - l)$ by

$$g_{n,l}(x) = g_n^{n-l}(x) := \frac{\prod_{i=0}^{l-1}(x^n - x^i)}{\prod_{i=0}^{l-1}(x^l - x^i)}.$$

Note that $g_{n,l}(x)$ is monic if $l(n - l) > 0$, while $g_{n,0}(x) = g_{n,n}(x) = 1$. We also put

$$g_{n,l}(x) = g_n^{n-l}(x) := 0 \quad \text{for } l < 0 \text{ or } l > n.$$

Then the number of $\mathbb{F}_{q^\nu}$-rational points of $G_{n,l} = G_n^{n-l}$ is equal to $g_{n,l}(q^\nu) = g_n^{n-l}(q^\nu)$. Let $\succ$ denote the lexicographic order on the set of pairs (l, d) of non-negative integers l and d:

$$(l, d) \succ (l', d') \iff l > l' \text{ or } (l = l' \text{ and } d > d').$$

By descending induction with respect to $\succ$, we define polynomials $\tau_{l,d}(x, y) \in \mathbb{Z}[x, y]$ as follows:

$$\tau_{l,d}(x, y) := \begin{cases} 0 & \text{if } l > n \text{ or } d > l, \\ g_{n,l}(x) & \text{if } d = l \leqslant n, \end{cases} \tag{2.4}$$

and, for $d < l \leqslant n$, by

$$\tau_{l,d}(x, y) := \sum_{u=l}^{2l-d} \tau_{2l-d,u}(x, y) \cdot g_{u,l}(y) - \sum_{t=d+1}^{l} \tau_{l,t}(x, y) \cdot g_{n-2l+t,t-d}(y). \tag{2.5}$$

Finally, for positive integers l and c with $l+c<n$, we put

$$N_l^c(x,y) := \sum_{d=0}^{l} \tau_{l,d}(x,y)\cdot g_{n-2l+d}^c(y) \in \mathbb{Z}[x,y]. \tag{2.6}$$

The main result of this section is as follows:

Theorem 2.2. *The polynomial $N_l^c(x,y)$ is monic of degree $(l+c)(n-l-c)$ with respect to the variable y, and the number of $\mathbb{F}_{(rs)^\nu}$-rational points of $X=X[r,s]_l^c$ is equal to $N_l^c(rs,(rs)^\nu)$.*

Theorem 2.2 provides us with an algorithm to calculate the Betti numbers of X by (1.5). From Proposition 2.1 and the fact that $N_l^c(x,y)$ is monic with respect to y, we obtain the following:

Corollary 2.3. *The Frobenius incidence variety X is geometrically irreducible.*

Thus the proof of Proposition 1.1 and Theorem 1.3 will be completed by Theorem 2.2.

For the proof of Theorem 2.2, we let q be a power of p by a positive integer, and define locally-closed reduced subvarieties $T_{l,d}$ of $G_{n,l}$ over $\mathbb{F}_p$ by the property that

$$T_{l,d}(F) = \left\{ L\in G_{n,l}(F) \;\middle|\; \dim\left(L\cap L^q\right)=d \right\} \tag{2.7}$$

should hold for any field F of characteristic p. First we prove the following:

Proposition 2.4. *For any pair (l,d) of non-negative integers l and d, the number of $\mathbb{F}_{q^\nu}$-rational points of $T_{l,d}$ is equal to $\tau_{l,d}(q,q^\nu)$.*

Proof. We proceed by descending induction on (l,d) with respect to $\succ$. By definition, we have $T_{l,d}(\mathbb{F}_{q^\nu})=\emptyset$ for any $\nu\in\mathbb{Z}_{>0}$ if $l>n$ or $d>l$. Since $L=L^q$ is equivalent to the condition that L be $\mathbb{F}_q$-rational, we have

$$T_{l,l}(\mathbb{F}_{q^\nu}) = G_{n,l}(\mathbb{F}_q) \quad \text{for all } \nu\in\mathbb{Z}_{>0}.$$

Thus $|T_{l,d}(\mathbb{F}_{q^\nu})|=\tau_{l,d}(q,q^\nu)$ holds for any (l,d) with $l>n$ or $d\geqslant l$ by (2.4). Suppose that $d<l\leqslant n$ and that $|T_{l',d'}(\mathbb{F}_{q^\nu})|=\tau_{l',d'}(q,q^\nu)$ holds for any (l',d') with $(l',d')\succ(l,d)$. We count the number of the elements of the finite set

$$\mathcal{P}_{l,d} := \left\{ (L,M)\in G_{n,l}(\mathbb{F}_{q^\nu})\times G_{n,2l-d}(\mathbb{F}_{q^\nu}) \;\middle|\; L+L^q\subset M \right\}$$

in two ways. If $(L,M)\in\mathcal{P}_{l,d}$, then we have $d\leqslant \dim(L\cap L^q)\leqslant l$. If $L\in T_{l,t}(\mathbb{F}_{q^\nu})$ with $d\leqslant t\leqslant l$, then $\dim(L+L^q)=2l-t$ holds and the number of $M\in G_{n,2l-d}(\mathbb{F}_{q^\nu})$ containing $L+L^q$ is equal to $g_{n-2l+t,t-d}(q^\nu)$. Hence we have

$$|\mathcal{P}_{l,d}| = \sum_{t=d}^{l} \left|T_{l,t}(\mathbb{F}_{q^\nu})\right|\cdot g_{n-2l+t,t-d}\left(q^\nu\right). \tag{2.8}$$

On the other hand, a pair $(L,M)\in G_{n,l}(\mathbb{F}_{q^\nu})\times G_{n,2l-d}(\mathbb{F}_{q^\nu})$ satisfies $L+L^q\subset M$ if and only if $L^q\subset M\cap M^q$, or equivalently, if and only if $L\subset M\cap M^{1/q}$. Note that $M^{1/q}$ is also $\mathbb{F}_{q^\nu}$-rational. If $L^q\subset M\cap M^q$

holds, then we have $l \leqslant \dim(M \cap M^q) \leqslant 2l - d$. If $\dim(M \cap M^q) = u$ with $l \leqslant u \leqslant 2l - d$, then the number of $L \in G_{n,l}(\mathbb{F}_{q^\nu})$ contained in $M \cap M^{1/q}$ is equal to $g_{u,l}(q^\nu)$. Hence we have

$$|\mathcal{P}_{l,d}| = \sum_{u=l}^{2l-d} \bigl|T_{2l-d,u}(\mathbb{F}_{q^\nu})\bigr| \cdot g_{u,l}\bigl(q^\nu\bigr). \tag{2.9}$$

Comparing (2.8) and (2.9), we obtain

$$\begin{aligned} \bigl|T_{l,d}(\mathbb{F}_{q^\nu})\bigr| \cdot g_{n-2l+d,0}\bigl(q^\nu\bigr) &= \sum_{u=l}^{2l-d} \bigl|T_{2l-d,u}(\mathbb{F}_{q^\nu})\bigr| \cdot g_{u,l}\bigl(q^\nu\bigr) - \sum_{t=d+1}^{l} \bigl|T_{l,t}(\mathbb{F}_{q^\nu})\bigr| \cdot g_{n-2l+t,t-d}\bigl(q^\nu\bigr) \\ &= \tau_{l,d}\bigl(q, q^\nu\bigr) \end{aligned}$$

by the induction hypothesis and the defining equality (2.5). If $n - 2l + d < 0$, then $g_{n-2l+d,0}(x) = 0$ and hence we have $\tau_{l,d}(q, q^\nu) = 0$, while we have $T_{l,d}(\mathbb{F}_{q^\nu}) = \emptyset$ because $L \in T_{l,d}(\mathbb{F}_{q^\nu})$ would imply $\dim(L + L^q) > n$. If $n - 2l + d \geqslant 0$, then we have $|T_{l,d}(\mathbb{F}_{q^\nu})| = \tau_{l,d}(q, q^\nu)$ because $g_{n-2l+d,0}(q^\nu) = 1$. Therefore $|T_{l,d}(\mathbb{F}_{q^\nu})| = \tau_{l,d}(q, q^\nu)$ is proved for any (l, d). □

Next we put

$$\delta(l, d) := (l - d)(n - l + d),$$

and prove the following:

Proposition 2.5. *Consider the following condition*:

$$\mathrm{C}(l, d): \quad \max(0, 2l - n) \leqslant d \leqslant l \leqslant n.$$

If $\mathrm{C}(l, d)$ is false, then $\tau_{l,d}(x, y) = 0$. If $\mathrm{C}(l, d)$ is true, then $\tau_{l,d}(x, y)$ is non-zero and of degree $\delta(l, d)$ with respect to y. If $\mathrm{C}(l, d)$ is true and $\delta(l, d) > 0$, then $\tau_{l,d}(x, y)$ is monic with respect to y.

Proof. First remark that, if $a(x, y) \in \mathbb{Z}[x, y]$ satisfies $a(q, q^\nu) = 0$ for any prime powers q and any positive integers ν, then we have $a(x, y) = 0$.

If $\mathrm{C}(l, d)$ is false, then $T_{l,d}$ is an empty variety for any q by definition, and hence $\tau_{l,d}(x, y) = 0$ by Proposition 2.4.

We prove the assertion

$$\mathrm{S}(l, d) \quad \begin{cases} \mathrm{C}(l, d) \;\Rightarrow\; \tau_{l,d}(x, y) \neq 0 \text{ and } \deg_y \tau_{l,d}(x, y) = \delta(l, d), \\ \mathrm{C}(l, d) \text{ and } \delta(l, d) > 0 \;\Rightarrow\; \tau_{l,d}(x, y) \text{ is monic with respect to } y, \end{cases}$$

by descending induction on (l, d) with respect to the order $\succ$. If $\mathrm{C}(l, d)$ is false, then $\mathrm{S}(l, d)$ holds vacuously. Hence we can assume that $\mathrm{S}(l', d')$ holds for any (l', d') with $(l', d') \succ (l, d)$, and that $\mathrm{C}(l, d)$ is true. If $d = l$, then $\mathrm{S}(l, d)$ holds because $\tau_{l,l}(x, y) = g_{n,l}(x)$ is a non-zero constant with respect to y. We assume that $d < l$. Note that now we have $n - l + d \geqslant l > d$ and $\delta(l, d) > 0$. First we study the second summation of (2.5). For t with $d + 1 \leqslant t \leqslant l$, we have

$$\delta(l, t) + \deg g_{n-2l+t,t-d}(y) = \delta(l, d) - t(t - d) < \delta(l, d).$$

Hence, by the induction hypothesis, every term in the second summation is non-zero of degree $<\delta(l,d)$ with respect to y. Next we study the first summation of (2.5). Note that the condition $\mathrm{C}(2l-d,u)$ on u is

$$\max(0, 4l-2d-n) \leqslant u \leqslant 2l-d.$$

From the induction hypothesis, the non-zero terms in the first summation are

$$s_u := \tau_{2l-d,u}(x,y) \cdot g_{u,l}(y) \quad \text{with } \max(l, 4l-2d-n) \leqslant u \leqslant 2l-d.$$

By the equality

$$\delta(2l-d,u) + \deg g_{u,l}(y) = \delta(l,d) - (u-l)(u-4l+2d+n),$$

every non-zero term in the first summation is of degree $\leqslant \delta(l,d)$ with respect to y. Moreover there exists one and only one term of degree equal to $\delta(l,d)$, which is

$$\begin{cases} s_l = \tau_{2l-d,l}(x,y) \cdot g_{l,l}(y) & \text{if } l \geqslant 4l-2d-n, \\ s_{4l-2d-n} = \tau_{2l-d,4l-2d-n}(x,y) \cdot g_{4l-2d-n,l}(y) & \text{if } l < 4l-2d-n. \end{cases} \tag{2.10}$$

It remains to show that this term is monic with respect to y. In the case where $l \geqslant 4l-2d-n$, the term s_l is monic with respect to y, because $g_{l,l}(y) = 1$ and $\delta(2l-d,l) = \delta(l,d) > 0$. We consider the case $l < 4l-2d-n$. Note that

$$\delta_1 := \delta(2l-d, 4l-2d-n) = (n-2l+d)(2l-d) \geqslant 0,$$

and that $\delta_1 = 0$ holds if and only if $n-2l+d = 0$, because we have $2l-d > l \geqslant 0$. Suppose that $\delta_1 > 0$. Then $\tau_{2l-d,4l-2d-n}(x,y)$ is monic of degree δ_1 with respect to y by the induction hypothesis. If $\delta_1 = 0$, then we have

$$\tau_{2l-d,4l-2d-n}(x,y) = \tau_{n,n}(x,y) = g_{n,n}(x) = 1.$$

On the other hand, $g_{4l-2d-n,l}(y)$ is monic of degree >0 for $l < 4l-2d-n$. Therefore the term $s_{4l-2d-n}$ in the second case of (2.10) is also monic with respect to y. Thus the statement $\mathrm{S}(l,d)$ holds. □

Proof of Theorem 2.2. We show that $N_l^c(x,y)$ is monic of degree

$$\tilde{\delta}(l,c) := (l+c)(n-l-c)$$

with respect to y. We put

$$t_d := \tau_{l,d}(x,y) \cdot g_{n-2l+d}^c(y),$$

so that $N_l^c(x,y) = \sum_{d=0}^{l} t_d$. If $d < 2l-n+c$, then $t_d = 0$ because $g_{n-2l+d}^c(y) = 0$. Suppose that $d \geqslant 0$ satisfies $d \geqslant 2l-n+c$. Then we have

$$\delta(l,d) + \deg g_{n-2l+d}^c(y) = \tilde{\delta}(l,c) - d(d-2l+n-c) \leqslant \tilde{\delta}(l,c).$$

Hence every non-zero term t_d is of degree $\leqslant \tilde{\delta}(l,c)$ with respect to y, and there are at most two terms that are of degree equal to $\tilde{\delta}(l,c)$; namely,

$$t_0 = \tau_{l,0}(x,y) \cdot g_{n-2l}^c(y) \quad \text{and} \quad t_{2l-n+c} = \tau_{l,2l-n+c}(x,y) \cdot g_c^c(y).$$

If $2l-n+c<0$, then t_{2l-n+c} does not appear in the summation $\sum_{d=0}^{l} t_d$, and t_0 is non-zero and monic with respect to y because $\mathrm{C}(l,0)$ holds, $\delta(l,0)>0$ and $\deg g^c_{n-2l}(y)>0$. If $2l-n+c=0$, then $t_{2l-n+c}=t_0$ is non-zero and monic with respect to y because $\mathrm{C}(l,0)$ holds, $\delta(l,0)>0$ and $g^c_{n-2l}(y)=g^c_c(y)=1$. If $2l-n+c>0$, then $t_0=0$ because $g^c_{n-2l}(y)=0$, and the term $t_{2l-n+c}=\tau_{l,2l-n+c}(x,y)$, which appears in the summation since $2l-n+c<l$, is non-zero and monic with respect to y because $\mathrm{C}(l,2l-n+c)$ holds and $\delta(l,2l-n+c)>0$. Thus, in each case, there exists one and only one term that is non-zero of degree $\tilde{\delta}(l,c)$, and this term is monic with respect to y. Hence the assertion is proved.

Finally we prove that the number of $\mathbb{F}_{(rs)^\nu}$-rational points of $X=X[r,s]^c_l$ is equal to $N^c_l(rs,(rs)^\nu)$. For simplicity, we put $q:=rs$. By the property (2.1) of X, it is enough to show that the number of the pairs $(L,M)\in G_{n,l}(\mathbb{F}_{q^\nu})\times G^c_n(\mathbb{F}_{q^\nu})$ satisfying $L+L^q\subset M^s$ is equal to $N^c_l(q,q^\nu)$. Note that $G_{n,l}(\mathbb{F}_{q^\nu})$ is the disjoint union of the finite sets $T_{l,d}(\mathbb{F}_{q^\nu})$ over d with $0\leqslant d\leqslant l$. If $L\in G_{n,l}(\mathbb{F}_{q^\nu})$ is contained in $T_{l,d}(\mathbb{F}_{q^\nu})$, then $L+L^q$ is of dimension $2l-d$ and hence the number of $M'\in G^c_n(\mathbb{F}_{q^\nu})$ containing $L+L^q$ is $g^c_{n-2l+d}(q^\nu)$. Because $M\mapsto M^s$ is a bijection from $G^c_n(\mathbb{F}_{q^\nu})$ to itself, the number of the pairs is equal to the sum of $\tau_{l,d}(q,q^\nu)\cdot g^c_{n-2l+d}(q^\nu)$ over d with $0\leqslant d\leqslant l$ by Proposition 2.5. Thus we have $|X(\mathbb{F}_{q^\nu})|=N^c_l(q,q^\nu)$ by the definition (2.6). □

The following is useful in the computation of $N^c_l(x,y)$:

Corollary 2.6. *We have $\tau_{l,d}(x,y)=\tau_{n-l,n-2l+d}(x,y)$ for any (l,d).*

Proof. We choose an inner product $V\times V\to\mathbb{F}_p$ defined over $\mathbb{F}_p$, and denote by $L^\perp\subset V_F$ the orthogonal complement of a linear subspace $L\subset V_F$. Then $L\mapsto L^\perp$ induces an isomorphism $G_{V,l}\xrightarrow{\sim}G_{V,n-l}$. Since $(L^q)^\perp=(L^\perp)^q$ and $L^\perp\cap(L^\perp)^q=(L+L^q)^\perp$, this isomorphism induces a bijection from $T_{l,d}(\mathbb{F}_{q^\nu})$ to $T_{n-l,n-2l+d}(\mathbb{F}_{q^\nu})$. Thus the equality follows from Proposition 2.4 and the remark at the beginning of the proof of Proposition 2.5. □

Example 2.7. We have

$$N^1_1(x,y)=g_{n,1}(y)\cdot g^1_{n-2}(y)+g_{n,1}(x)\cdot\big(g^1_{n-1}(y)-g^1_{n-2}(y)\big),$$

and hence the Betti numbers of $X[r,s]^1_1$ in Example 1.4 are obtained.

Example 2.8. We have

$$\begin{aligned}\tau_{2,1}(x,y)&=\tau_{n-2,n-3}(x,y)\\&=g_{n,n-1}(y)-g_{n,n-1}(x)+g_{n,n-1}(x)\cdot g_{n-1,n-2}(y)-g_{n,n-2}(x)\cdot g_{2,1}(y),\end{aligned}$$

and hence

$$\begin{aligned}N^2_2(x,y)=\;&g^2_n(y)\cdot g^2_{n-4}(y)+g^1_n(y)\cdot g^2_{n-3}(y)+g^1_n(x)\cdot g^1_{n-1}(y)\cdot g^2_{n-3}(y)\\&-g^1_n(y)\cdot g^2_{n-4}(y)-g^1_n(x)\cdot g^1_{n-1}(y)\cdot g^2_{n-4}(y)+g^2_n(x)\cdot g^2_{n-2}(y)\\&-g^2_n(x)\cdot g^1_2(y)\cdot g^2_{n-3}(y)-g^1_n(x)\cdot g^2_{n-3}(y)+g^2_n(x)\cdot g^1_2(y)\cdot g^2_{n-4}(y)\\&-g^2_n(x)\cdot g^2_{n-4}(y)+g^1_n(x)\cdot g^2_{n-4}(y).\end{aligned}$$

For instance, consider the case where $n=7$. Then the Betti numbers of the 12-dimensional Frobenius incidence variety $X[r,s]_2^2$ are as follows, where $q:=rs$:

$$\begin{aligned}
b_0 = b_{24}&: \quad 1,\\
b_2 = b_{22}&: \quad 2,\\
b_4 = b_{20}&: \quad 5,\\
b_6 = b_{18}&: \quad q^6+q^5+q^4+q^3+q^2+q+8,\\
b_8 = b_{16}&: \quad 2\bigl(q^6+q^5+q^4+q^3+q^2+q\bigr)+12,\\
b_{10} = b_{14}&: \quad 3\bigl(q^6+q^5+q^4+q^3+q^2+q\bigr)+14,\\
b_{12}&: \quad q^{10}+q^9+2q^8+2q^7+6q^6+6q^5+6q^4+5q^3+5q^2+4q+16.
\end{aligned}$$

Remark 2.9. The fact that $N_l^c(x,y)$ should be palindromic with respect to y helps us in checking the computation of $N_l^c(x,y)$.

As a simple corollary of Propositions 2.4 and 2.5, we obtain the following. Let κ denote the function field of X. By the generic point of X, we mean the pair (L_η, M_η) of κ-rational linear subspaces corresponding to

$$\operatorname{Spec}\kappa \to X \hookrightarrow G_{n,l} \times G_n^c,$$

where $\operatorname{Spec}\kappa \to X$ is the canonical morphism.

Proposition 2.10. *Let (L_η, M_η) be the generic point of X.*

(1) *If $2l+c \geqslant n$, then $L_\eta + L_\eta^{rs} = M_\eta^s$. If $2l+c \leqslant n$, then the projection $X \to G_{n,l}$ is surjective and hence $L_\eta + L_\eta^{rs}$ is of dimension $2l$.*
(2) *If $l+2c \geqslant n$, then $M_\eta \cap M_\eta^{rs} = L_\eta^r$. If $l+2c \leqslant n$, then the projection $X \to G_n^c$ is surjective and hence $M_\eta \cap M_\eta^{rs}$ is of dimension $n-2c$.*

Proof. We put $q:=rs$ again. The function $d_L : (L,M) \mapsto \dim(L+L^q)$ is lower semi-continuous and bounded by $n-c$ from above on X. If $2l+c \geqslant n$, then $\mathrm{C}(l, 2l-n+c)$ is true and $\delta(l, 2l-n+c) > 0$. Therefore the set $T_{l,2l-n+c}(\mathbb{F}_{q^\nu})$ is non-empty for a sufficiently large ν. Hence d_L attains $n-c$ on X, and thus $\dim(L_\eta + L_\eta^q) = n-c$. Therefore we have $L_\eta + L_\eta^q = M_\eta^s$. Let κ_γ denote the function field of $G_{n,l}$. If $2l+c \leqslant n$, then the generic point $L_\gamma \in G_{n,l}(\kappa_\gamma)$ satisfies $\dim(L_\gamma + L_\gamma^q) = 2l \leqslant n-c$. There exists a κ_γ-valued point $N_\gamma \in G_n^c(\kappa_\gamma)$ such that $L_\gamma + L_\gamma^q \subset N_\gamma$, and hence $(L_\gamma, N_\gamma^{1/s})$ is a $\kappa_\gamma^{1/s}$-valued point of X. Thus the assertion (1) is proved.

The assertion (2) is proved in the dual way. $\square$

3. Unirationality

In this section, we prove Theorem 1.5.

Note that the purely inseparable morphisms

$$\phi^{(s)} \times \mathrm{id} : G_{V,l} \times G_V^c \to G_{V,l} \times G_V^c \quad \text{and} \quad \mathrm{id} \times \phi^{(r)} : G_{V,l} \times G_V^c \to G_{V,l} \times G_V^c$$

induce purely inseparable surjective morphisms

$$X[rs,1] \to X[r,s] \quad \text{and} \quad X[1,rs] \to X[r,s]$$

defined over $\mathbb{F}_p$. Hence it is enough to prove that $X[q,1]$ is purely-inseparably unirational over $\mathbb{F}_p$ for any power q of p. We prove this fact by induction on $2l+c$.

Suppose that $2l+c \leqslant n$. We show that $X[q,1]$ is rational over $\mathbb{F}_p$, and hence, *a fortiori*, purely-inseparably unirational over $\mathbb{F}_p$. For the generic point (L_η, M_η) of $X[q,1]$, we have $\dim(L_\eta + L_\eta^q) = 2l$ by Proposition 2.10(1). We fix an $\mathbb{F}_p$-rational linear subspace $K \subset V$ of dimension $2l$. Then there exist a non-empty open subset $\mathcal{U}$ of $G_{V,l}$ and a morphism

$$\alpha : \mathcal{U} \to GL(V)$$

defined over $\mathbb{F}_p$ such that, for any field F of characteristic p and any F-valued point $L \in \mathcal{U}(F)$, we have $\dim(L+L^q) = 2l$ and $\alpha(L) \in GL(V_F)$ induces an isomorphism from $K \otimes F$ to $L+L^q$. Let $M/K \mapsto M$ denote the natural embedding $G^c_{V/K} \hookrightarrow G^c_V$. Then the morphism

$$\mathcal{U} \times G^c_{V/K} \to G_{V,l} \times G^c_V$$

given by $(L, M/K) \mapsto (L, \alpha(L)(M))$ is a birational map defined over $\mathbb{F}_p$ from the rational variety $\mathcal{U} \times G^c_{V/K}$ to $X[q,1]$, with the inverse rational map being given by

$$(L, M) \mapsto \bigl(L, \alpha(L)^{-1}(M)/K\bigr).$$

Suppose that $2l+c > n$. We put

$$l' := 2l + c - n \quad \text{and} \quad c' := n - l.$$

Then we have $l' > 0$, $c' > 0$ and $l' + c' < n$. We show that $X := X[q,1]^c_l$ is birational over $\mathbb{F}_p$ to $X' := X[1,q]^{c'}_{l'}$. We denote by κ and κ' the function fields of X and X', respectively. Note that

$$X(F) = \bigl\{(L, M) \in G_{n,l}(F) \times G^c_n(F) \bigm| L + L^q \subset M\bigr\},$$
$$X'(F) = \bigl\{\bigl(L', M'\bigr) \in G_{n,2l+c-n}(F) \times G_{n,l}(F) \bigm| L' \subset M' \cap M'^q\bigr\}.$$

By Proposition 2.10(1), the generic point (L_η, M_η) of X satisfies $L_\eta + L_\eta^q = M_\eta$, and hence $\dim(L_\eta \cap L_\eta^q) = 2l + c - n$. Therefore we have $(L_\eta \cap L_\eta^q, L_\eta) \in X'(\kappa)$, and hence

$$(L, M) \mapsto \bigl(L \cap L^q, L\bigr)$$

defines a rational map $\rho : X \dashrightarrow X'$ defined over $\mathbb{F}_p$. On the other hand, we have

$$l' + 2c' = n + c > n.$$

By Proposition 2.10(2), the generic point (L'_η, M'_η) of X' satisfies $M'_\eta \cap M'^q_\eta = L'_\eta$, and hence

$$\dim\bigl(M'_\eta + M'^q_\eta\bigr) = 2\bigl(n - c'\bigr) - l' = n - c.$$

Therefore we have $(M'_\eta, M'_\eta + M'^q_\eta) \in X(\kappa')$, and hence

$$\bigl(L', M'\bigr) \mapsto \bigl(M', M' + M'^q\bigr)$$

defines a rational map $\rho' : X' \dashrightarrow X$ defined over $\mathbb{F}_p$. Note that $\rho'(\rho(L_\eta, M_\eta))$ is defined and equal to (L_η, M_η). Note also that $\rho(\rho'(L'_\eta, M'_\eta))$ is defined and equal to (L'_η, M'_η). Hence X and X' are birational over $\mathbb{F}_p$. Since

$$2l' + c' = 2l + c - (n - l - c) < 2l + c,$$

the induction hypothesis implies that X' is purely-inseparably unirational over $\mathbb{F}_p$. Therefore X is also purely-inseparably unirational over $\mathbb{F}_p$. $\square$

Remark 3.1. We have established the facts that $X[q,1]^c_l$ is rational over $\mathbb{F}_p$ for $2l + c \leqslant n$, and that $X[1,q]^c_l$ is rational over $\mathbb{F}_p$ for $l + 2c \leqslant n$.

4. Intersection pairing

In this section, we calculate the intersections of the subvarieties Σ_Λ defined by (1.6) in the Chow ring of $X = X[r,s]^c_l$.

For a smooth projective variety Y of dimension m, we denote by $A^k(Y) = A_{m-k}(Y)$ the Chow group of rational equivalence classes of algebraic cycles on Y with codimension k defined over an algebraic closure of the base field, and by $A(Y) = \bigoplus A^k(Y)$ the Chow ring of Y.

In order to state our main result, we need to define a homomorphism $\tilde{\psi}$. Let W be a w-dimensional linear space, and let

$$S_{W,l} \to G_{W,l} = G_{w,l}$$

denote the universal subbundle of $W \otimes \mathcal{O}$ over $G_{W,l}$. Let $x_1, \dots, x_l$ be the formal Chern roots of the total Chern class $c(S^\vee_{W,l})$ of the dual vector bundle $S^\vee_{W,l}$:

$$c\big(S^\vee_{W,l}\big) = (1 + x_1) \cdot \cdots \cdot (1 + x_l).$$

Then we have a natural homomorphism

$$\psi_{w,l} : \mathbb{Z}[\![x_1, \dots, x_l]\!]^{\mathfrak{S}_l} \to A(G_{W,l})$$

from the ring of symmetric power series in variables $x_1, \dots, x_l$ with $\mathbb{Z}$-coefficients to the Chow ring $A(G_{W,l})$ of $G_{W,l}$. Let U be a u-dimensional linear space, and let

$$Q^c_U \to G^c_U = G^c_u$$

denote the universal quotient bundle of $U \otimes \mathcal{O}$ over G^c_U. Let $y_1, \dots, y_c$ be the formal Chern roots of the total Chern class $c(Q^c_U)$:

$$c\big(Q^c_U\big) = (1 + y_1) \cdot \cdots \cdot (1 + y_c).$$

Then we have a natural homomorphism

$$\psi^c_u : \mathbb{Z}[\![y_1, \dots, y_c]\!]^{\mathfrak{S}_c} \to A\big(G^c_U\big).$$

Composing $\psi_{w,l} \otimes \psi^c_u$ and the natural homomorphism

$$A(G_{W,l}) \otimes A\big(G^c_U\big) \to A\big(G_{w,l} \times G^c_u\big),$$

we obtain a homomorphism

$$\tilde{\psi} : \mathbb{Z}[\![x_1, \ldots, x_l, y_1, \ldots, y_c]\!]^{\mathfrak{S}_l \times \mathfrak{S}_c} \to A\bigl(G_{w,l} \times G_u^c\bigr)$$

from the ring of $\mathfrak{S}_l \times \mathfrak{S}_c$-symmetric power series to $A(G_{w,l} \times G_u^c)$.

Let Λ and Λ' be $\mathbb{F}_{rs}$-rational linear subspaces of $V_{\mathbb{F}} := V \otimes \mathbb{F}_{rs}$. We consider the intersection of the subvarieties Σ_Λ and $\Sigma_{\Lambda'}$ of X in $A(X)$. We put

$$m := \dim\bigl(\Lambda \cap \Lambda'\bigr) \quad \text{and} \quad k := n - \dim\bigl(\Lambda + \Lambda'\bigr).$$

Then we have

$$e := \dim \Sigma_\Lambda + \dim \Sigma_{\Lambda'} - \dim X = (l-c)(c-l+m-k),$$

and the intersection of Σ_Λ and $\Sigma_{\Lambda'}$ in $A(X)$ is an element of $A_e(X)$. We put

$$\Upsilon := \Lambda \cap \Lambda' \quad \text{and} \quad \Theta := V_{\mathbb{F}}/\bigl(\Lambda + \Lambda'\bigr)^r.$$

Since $\Sigma_\Lambda = G_{\Lambda,l} \times G^c_{V_{\mathbb{F}}/\Lambda^r}$ and $\Sigma_{\Lambda'} = G_{\Lambda',l} \times G^c_{V_{\mathbb{F}}/\Lambda'^r}$, the scheme-theoretic intersection of Σ_Λ and $\Sigma_{\Lambda'}$ is the smooth subscheme

$$\Gamma := G_{\Upsilon,l} \times G^c_\Theta \cong G_{m,l} \times G^c_k.$$

Then the intersection of Σ_Λ and $\Sigma_{\Lambda'}$ in $A_e(X)$ is localized in $A_e(\Gamma) = A^d(\Gamma)$, where

$$d := \dim \Gamma - e = kl + mc - 2lc.$$

The following is the main result of this section:

Theorem 4.1. *Let Λ and Λ' be as above. Then the intersection of Σ_Λ and $\Sigma_{\Lambda'}$ in $A(X)$ is equal to the image of the codimension d part of*

$$\tilde{\psi}(f) \in A(\Gamma)$$

by the push-forward homomorphism $A_e(\Gamma) \to A_e(X)$, where f is the $\mathfrak{S}_l \times \mathfrak{S}_c$-symmetric power series

$$\frac{\prod_{i=1}^l (1+x_i)^k \cdot \prod_{j=1}^c (1+y_j)^m}{\prod_{i=1}^l \prod_{j=1}^c (1+rx_i+y_j)(1+x_i+sy_j)},$$

and $\tilde{\psi}$ is the homomorphism to $A(\Gamma) = A(G_{m,l} \times G_k^c)$ defined above.

Proof. We denote by $\mathcal{T}(Y) \to Y$ the tangent bundle of a smooth variety Y. Note that the tangent bundle of a Grassmannian variety is the tensor product of the dual of the universal subbundle and the universal quotient bundle.

Let W and W' be linear subspaces of $V_{\mathbb{F}}$ with dimension w and w', respectively, such that $W \subset W'$, and let $i : G_{W,l} \hookrightarrow G_{W',l}$ denote the natural embedding. For the universal subbundles $S_{W,l} \to G_{W,l}$ and $S_{W',l} \to G_{W',l}$, we have

$$i^* S^\vee_{W',l} = S^\vee_{W,l}. \tag{4.1}$$

We denote by $Q_W^{w-l} \to G_{W,l}$ and $Q_{W'}^{w'-l} \to G_{W',l}$ the universal quotient bundles. Then we have an exact sequence

$$0 \to Q_W^{w-l} \to i^* Q_{W'}^{w'-l} \to W'/W \otimes \mathcal{O} \to 0$$

of vector bundles over $G_{W,l}$. Therefore we have the following equality in $A(G_{W,l})$:

$$i^* c\bigl(\mathcal{T}(G_{W',l})\bigr) = c\bigl(\mathcal{T}(G_{W,l})\bigr) \cdot c\bigl(S_{W,l}^{\vee}\bigr)^{w'-w}. \tag{4.2}$$

Let $V_{\mathbb{F}} \to U = V_{\mathbb{F}}/K$ and $V_{\mathbb{F}} \to U' = V_{\mathbb{F}}/K'$ be linear quotient spaces of $V_{\mathbb{F}}$ such that $K' \subset K$. We put $u := \dim U$ and $u' := \dim U'$. Let $j : G_U^c \hookrightarrow G_{U'}^c$ denote the natural embedding. For the universal quotient bundles $Q_U^c \to G_U^c$ and $Q_{U'}^c \to G_{U'}^c$, we have

$$j^* Q_{U'}^c = Q_U^c. \tag{4.3}$$

By the argument dual to the above, we obtain the following equality in $A(G_U^c)$:

$$j^* c\bigl(\mathcal{T}\bigl(G_{U'}^c\bigr)\bigr) = c\bigl(\mathcal{T}\bigl(G_U^c\bigr)\bigr) \cdot c\bigl(Q_U^c\bigr)^{u'-u}. \tag{4.4}$$

We consider the vector bundle

$$\mathcal{X} := \frac{\mathcal{T}(X)|_{\Gamma}}{\mathcal{T}(\Sigma_{\Lambda})|_{\Gamma} + \mathcal{T}(\Sigma_{\Lambda'})|_{\Gamma}}$$

of rank d over Γ. By the excess intersection formula [10, p. 102], the intersection of Σ_{Λ} and $\Sigma_{\Lambda'}$ in $A(X)$ is equal to the image of the top Chern class $c_d(\mathcal{X}) \in A^d(\Gamma) = A_e(\Gamma)$ of $\mathcal{X}$ by the push-forward homomorphism $A_e(\Gamma) \to A_e(X)$. Hence it is enough to show that the total Chern class $c(\mathcal{X}) \in A(\Gamma)$ of $\mathcal{X}$ is equal to $\tilde{\psi}(f)$. From the exact sequence

$$0 \to \mathcal{T}(\Gamma) \to \mathcal{T}(\Sigma_{\Lambda})|_{\Gamma} \oplus \mathcal{T}(\Sigma_{\Lambda'})|_{\Gamma} \to \mathcal{T}(\Sigma_{\Lambda})|_{\Gamma} + \mathcal{T}(\Sigma_{\Lambda'})|_{\Gamma} \to 0$$

and Proposition 2.1, we have

$$c(\mathcal{X}) = \frac{c(\mathcal{T}(G_{V,l} \times G_V^c)|_{\Gamma}) \cdot c(\mathcal{T}(\Gamma))}{c(\mathcal{F}|_{\Gamma}) \cdot c(\mathcal{T}(\Sigma_{\Lambda})|_{\Gamma}) \cdot c(\mathcal{T}(\Sigma_{\Lambda'})|_{\Gamma})}.$$

We put

$$\lambda := \dim \Lambda \quad \text{and} \quad \lambda' := \dim \Lambda'.$$

By (4.1)–(4.4), we have the following equalities in $A(\Gamma)$:

$$\begin{aligned}
c\bigl(\mathcal{T}\bigl(G_{V,l} \times G_V^c\bigr)\big|_{\Gamma}\bigr) &= c\bigl(\mathcal{T}(\Gamma)\bigr) \cdot \bigl(c\bigl(S_{\Upsilon,l}^{\vee}\bigr)^{n-m} \otimes c\bigl(Q_{\Theta}^c\bigr)^{n-k}\bigr), \\
c\bigl(\mathcal{T}(\Sigma_{\Lambda})|_{\Gamma}\bigr) &= c\bigl(\mathcal{T}(\Gamma)\bigr) \cdot \bigl(c\bigl(S_{\Upsilon,l}^{\vee}\bigr)^{\lambda-m} \otimes c\bigl(Q_{\Theta}^c\bigr)^{n-k-\lambda}\bigr), \\
c\bigl(\mathcal{T}(\Sigma_{\Lambda'})|_{\Gamma}\bigr) &= c\bigl(\mathcal{T}(\Gamma)\bigr) \cdot \bigl(c\bigl(S_{\Upsilon,l}^{\vee}\bigr)^{\lambda'-m} \otimes c\bigl(Q_{\Theta}^c\bigr)^{n-k-\lambda'}\bigr).
\end{aligned}$$

Here $c(S_{\Upsilon,l}^{\vee})^{\mu} \otimes c(Q_{\Theta}^c)^{\nu} \in A(G_{\Upsilon,l}) \otimes A(G_{\Theta}^c)$ is identified with its image in $A(\Gamma) = A(G_{\Upsilon,l} \times G_{\Theta}^c)$. Since $\lambda + \lambda' = m + n - k$, we have

$$c(\mathcal{X}) = c(\mathcal{F}|_{\Gamma})^{-1} \cdot \bigl(c\bigl(S_{\Upsilon,l}^{\vee}\bigr)^{k} \otimes c\bigl(Q_{\Theta}^c\bigr)^{m}\bigr).$$

By the definition (2.2) of $\mathcal{F}$ and (4.1) and (4.3), we have

$$\mathcal{F}|_\Gamma = \bigl(\phi^{(r)*} S^\vee_{\Upsilon,l} \otimes Q^c_\Theta\bigr) \oplus \bigl(S^\vee_{\Upsilon,l} \otimes \phi^{(s)*} Q^c_\Theta\bigr),$$

where pr^* is omitted. Note that

$$c\bigl(\phi^{(r)*} S^\vee_{\Upsilon,l}\bigr) = \prod_{i=1}^{l} (1 + r x_i) \quad \text{in } A(G_{\Upsilon,l}) \quad \text{and} \quad c\bigl(\phi^{(s)*} Q^c_\Theta\bigr) = \prod_{j=1}^{c} (1 + s y_j) \quad \text{in } A\bigl(G^c_\Theta\bigr).$$

Hence $c(\mathcal{F}|_\Gamma) \in A(\Gamma)$ is the image of

$$\prod_{i=1}^{l} \prod_{j=1}^{c} (1 + r x_i + y_j)(1 + x_i + s y_j) \in \mathbb{Z}[\![x_1, \dots, x_l, y_1, \dots, y_c]\!]^{\mathfrak{S}_l \times \mathfrak{S}_c}$$

by the homomorphism $\tilde{\psi}$. Therefore we have $c(\mathcal{X}) = \tilde{\psi}(f)$ in $A(\Gamma)$. $\square$

When $l = c$, the intersection number of Σ_Λ and $\Sigma_{\Lambda'}$ in $X[r,s]^l_l$ is defined and equal to the degree of the A_0-component of $\tilde{\psi}(f) \in A(\Gamma)$. When $l = c = 1$, we have $S^\vee_{\Upsilon,1} = \mathcal{O}(1)$ on $G_{\Upsilon,1} \cong \mathbb{P}^{m-1}$ and $Q^1_\Theta = \mathcal{O}(1)$ on $G^1_\Theta \cong \mathbb{P}^{k-1}$. Therefore we obtain the following:

Corollary 4.2. *Suppose that $l = c = 1$, and let Λ and Λ' be as above. Then the intersection number of Σ_Λ and $\Sigma_{\Lambda'}$ in $X[r,s]^1_1$ is equal to the coefficient of $x^{m-1} y^{k-1}$ in the power series*

$$(1 + r x + y)^{-1} (1 + x + s y)^{-1} (1 + x)^k (1 + y)^m \in \mathbb{Z}[\![x, y]\!].$$

5. The lattice $\mathcal{N}(X)$

In this section, we treat the case $l = c = 1$. We put $X := X[r,s]^1_1$ throughout this section.
We denote by $\operatorname{disc}\mathcal{N}(X)$ the discriminant of $\mathcal{N}(X)$. We put

$$f(n) := \frac{(rs)^n - 1}{rs - 1} = \bigl|\mathbb{P}^{n-1}(\mathbb{F}_{rs})\bigr|.$$

Theorem 5.1. *The rank of the lattice $\mathcal{N}(X)$ associated with $X = X[r,s]^1_1$ is equal to the middle Betti number $b_{2(n-2)}(X)$ of X. If $n > 3$, $\operatorname{disc}\mathcal{N}(X)$ is a divisor of $\min(r,s)^{(n-2)(f(n)-1)}$, while if $n = 3$, $\operatorname{disc}\mathcal{N}(X)$ is a divisor of $\min(r,s)^{f(3)+1}$.*

For the proof of Theorem 5.1, we fix notation. We write $[\Sigma_\Lambda] \in \mathcal{N}(X)$ and $[h_i] \in \mathcal{N}(X)$ for the rational equivalence classes of the algebraic cycles Σ_Λ and h_i modulo $\widetilde{\mathcal{N}}(X)^\perp$.

Let $\mathcal{P}_0$ denote the set of $\mathbb{F}_{rs}$-rational points of $\mathbb{P}_*(V)$, whose cardinality is $f(n)$. For a positive integer $k < n$, let $\mathcal{L}_k$ denote the set of k-dimensional $\mathbb{F}_{rs}$-rational linear subspaces of $V_{\mathbb{F}} := V \otimes \mathbb{F}_{rs}$. For $\Lambda \in \mathcal{L}_k$, we denote by $\mathbb{P}_*(\Lambda)$ the corresponding $(k-1)$-dimensional projective linear subspace of $\mathbb{P}_*(V)$ over $\mathbb{F}_{rs}$, and put

$$S(\Lambda) := \bigl\{P \in \mathcal{P}_0 \bigm| P \in \mathbb{P}_*(\Lambda)\bigr\}.$$

For $P \in \mathcal{P}_0$, let $\ell(P) \in \mathcal{L}_1$ denote the corresponding $\mathbb{F}_{rs}$-rational linear subspace of dimension 1.

We calculate the intersection numbers of the classes $[h_i]$ and $[\Sigma_\Lambda]$ in $\mathcal{N}(X)$. By Corollary 4.2, for $P \in \mathcal{P}_0$ and $\Lambda \in \mathcal{L}_k$, we have

$$\big([\Sigma_\Lambda], [\Sigma_{\ell(P)}]\big) = \begin{cases} (-s)^{n-k-1} & \text{if } P \in S(\Lambda), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

For $\Lambda \in \mathcal{L}_k$, the subvariety Σ_Λ is a Cartesian product of $\mathbb{P}_*(\Lambda) \subset \mathbb{P}_*(V)$ and $\mathbb{P}^*(V/\Lambda^r) \subset \mathbb{P}^*(V)$ with $\dim \mathbb{P}_*(\Lambda) = k-1$ and $\dim \mathbb{P}^*(V/\Lambda^r) = n-1-k$. Hence we have

$$\big([h_i], [\Sigma_\Lambda]\big) = \begin{cases} 1 & \text{if } i+k=n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Recall from Proposition 2.1 that $X \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V)$ is a subvariety of codimension 2 defined as the zero locus of the section $\tilde{\gamma}$ of the vector bundle $\mathcal{O}(r,1) \oplus \mathcal{O}(1,s)$ of rank 2. Hence the intersection numbers of the classes $[h_i]$ are

$$\big([h_i], [h_j]\big) = \begin{cases} s & \text{if } i+j=n-1, \\ 1+rs & \text{if } i+j=n, \\ r & \text{if } i+j=n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

We fix a point $B_0 \in \mathcal{P}_0$, and consider the following four submodules of $\mathcal{N}(X)$:

$$\begin{aligned} \mathcal{H} &:= \big\langle [h_1], \dots, [h_{n-1}] \big\rangle, \\ \mathcal{M} &:= \big\langle [\Sigma_{\ell(P)}] \bigm| P \in \mathcal{P}_0 \big\rangle, \\ \mathcal{M}_0 &:= \big\langle [\Sigma_{\ell(P)}] \bigm| P \in \mathcal{P}_0,\ P \neq B_0 \big\rangle, \\ \mathcal{M}_D &:= \big\langle [D_{\ell(P)}] \bigm| P \in \mathcal{P}_0,\ P \neq B_0 \big\rangle, \quad \text{where } [D_{\ell(P)}] := [\Sigma_{\ell(P)}] - [\Sigma_{\ell(B_0)}]. \end{aligned}$$

Here $\langle v_1, \dots, v_N \rangle$ denotes the submodule generated by $v_1, \dots, v_N$.

The following is elementary:

Lemma 5.2. *Let m be an integer $\geqslant 3$, and let u, v, t be indeterminants. Consider the $m \times m$ matrix $A(m,u,v,t) = (a_{ij})_{1 \leqslant i,j \leqslant m}$ defined by*

$$a_{ij} := \begin{cases} u & \text{if } i+j=m, \\ 1+uv & \text{if } i+j=m+1, \\ v & \text{if } i+j=m+2, \\ t & \text{if } i=j=m, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\det A(m,u,v,t) = (-1)^{[m/2]} \left(\frac{(uv)^{m+1}-1}{uv-1} + (-u)^{m-1} t \right),$$

where $[m/2]$ denotes the integer part of $m/2$.

Proof of Theorem 5.1. By the duality, we can assume that $s \leqslant r$.

If the cohomology class of $x \in \widetilde{\mathcal{N}}(X)$ is zero, then x is obviously contained in $\widetilde{\mathcal{N}}(X)^{\perp}$. Hence, by Example 1.4, the rank of $\mathcal{N}(X)$ is at most

$$b := b_{2(n-2)}(X) = n + f(n) - 2 = (n-1) + \bigl(|\mathcal{P}_0| - 1\bigr).$$

First assume that $n > 3$. We show that $\mathcal{N}(X)$ is of rank b, and that its discriminant divides $s^{(n-2)(f(n)-1)}$. Consider the submodule $\mathcal{H} + \mathcal{M}_0$ of $\mathcal{N}(X)$ generated by the b classes

$$[h_1], \dots, [h_{n-1}],\ [\Sigma_{\ell(P)}] \quad (P \in \mathcal{P}_0,\ P \neq B_0). \tag{5.4}$$

By (5.1), (5.2) and (5.3), the intersection matrix of these classes is

$$\widetilde{A} := \left[\begin{array}{c|c} A_{\mathcal{H}} & \begin{matrix} O \\ 1 \ \cdots \ 1 \end{matrix} \\ \hline \begin{matrix} & 1 \\ O & \vdots \\ & 1 \end{matrix} & (-s)^{n-2} I \end{array}\right],$$

where $A_{\mathcal{H}} := A(n-1, s, r, 0)$ is the intersection matrix of the classes $[h_i]$ and I is the identity matrix of size $f(n) - 1$. By Lemma 5.2, we have

$$\begin{aligned} \det \widetilde{A} &= \det A(n-1, s, r, t_0) \cdot \det\bigl((-s)^{n-2} I\bigr) \\ &= (-1)^{[(n-1)/2]} \cdot (-s)^{(n-2)(f(n)-1)} \neq 0, \end{aligned}$$

where $t_0 := -(f(n) - 1)/(-s)^{n-2}$. Thus $\mathcal{H} + \mathcal{M}_0$ is a lattice of rank b with the basis (5.4). Since $\operatorname{rank} \mathcal{N}(X) \leqslant b$, we conclude that $\operatorname{rank} \mathcal{N}(X) = b$ and that $\mathcal{N}(X)$ contains $\mathcal{H} + \mathcal{M}_0$ as a sublattice of finite index. Therefore $\operatorname{disc} \mathcal{N}(X)$ is a divisor of $\operatorname{disc}(\mathcal{H} + \mathcal{M}_0) = \pm s^{(n-2)(f(n)-1)}$.

For the case $n = 3$, we consider the submodule $\langle [h_1] \rangle + \mathcal{M}$. The intersection matrix of the generators $[h_1]$ and $[\Sigma_{\ell(P)}]$ $(P \in \mathcal{P}_0)$ of this submodule is the diagonal matrix of size $b = f(3) + 1$ with diagonal components $s, -s, \dots, -s$. Consequently, $\langle [h_1] \rangle + \mathcal{M}$ is a lattice of rank b with the discriminant $\pm s^{f(3)+1}$. Hence $\mathcal{N}(X)$ is a lattice of rank b containing $\langle [h_1] \rangle + \mathcal{M}$ as a sublattice of finite index, and $\operatorname{disc} \mathcal{N}(X)$ is a divisor of $s^{f(3)+1}$. □

Proof of Corollary 1.6. By Proposition 2.1, the subvariety $X \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V)$ is a smooth complete intersection of very ample divisors $D_1 \in |\mathcal{O}(r, 1)|$ and $D_2 \in |\mathcal{O}(1, s)|$. Hence, by Lefschetz hyperplane section theorem of Deligne [5] (see also [19]), the inclusion of X into $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$ induces isomorphisms of l-adic cohomology groups in degree $< \dim X$. On the other hand, Theorem 5.1 implies that the cycle map induces an isomorphism from $\mathcal{N}(X) \otimes \mathbb{Q}_l$ to the middle l-adic cohomology group of X. □

Remark 5.3. Theorem 5.1 implies that, if $r = 1$ or $s = 1$, then $\mathcal{N}(X[r, s]_1^1)$ is unimodular. Recall from Remark 3.1 that, if $r = 1$ or $s = 1$, then $X[r, s]_1^1$ is a rational variety.

Next we prove Theorem 1.7 on the primitive part $\mathcal{N}_{\mathrm{prim}}(X)$ of $\mathcal{N}(X)$.

Proof of Theorem 1.7. We use the notation in the proof of Theorem 5.1. Since

$$\det A_{\mathcal{H}} = \det A(n-1, s, r, 0) = \pm f(n) \neq 0,$$

the submodule $\mathcal{H}$ is a sublattice of $\mathcal{N}(X)$ with rank $n-1$. Therefore $\mathcal{N}_{\mathrm{prim}}(X) = \mathcal{H}^{\perp}$ is also a sublattice with

$$\operatorname{rank}\mathcal{N}_{\mathrm{prim}}(X) = b - (n-1) = f(n) - 1.$$

By (5.2), the classes

$$[\Sigma_{\Lambda}] - [\Sigma_{\Lambda'}] \quad \bigl(\Lambda, \Lambda' \in \mathcal{L}_k,\ k = 1, \dots, n-1\bigr) \tag{5.5}$$

are contained in $\mathcal{N}_{\mathrm{prim}}(X)$. In particular, we have $\mathcal{M}_D \subset \mathcal{N}_{\mathrm{prim}}(X)$. By (5.1), we have

$$\bigl([D_{\ell(P)}], [D_{\ell(P')}]\bigr) = \begin{cases} 2(-s)^{n-2} & \text{if } P = P', \\ (-s)^{n-2} & \text{if } P \neq P'. \end{cases}$$

Hence the intersection matrix A_D of the classes $[D_{\ell(P)}]$ $(P \in \mathcal{P}_0, P \neq B_0)$ is non-degenerate. Therefore $\mathcal{M}_D$ is of rank $f(n)-1$, and we have

$$\mathcal{M}_D \otimes \mathbb{Q} = \mathcal{N}_{\mathrm{prim}}(X) \otimes \mathbb{Q}. \tag{5.6}$$

The symmetric matrix A_D multiplied by $(-1)^n$ defines a positive-definite quadratic form. Hence $[-1]^n\mathcal{N}_{\mathrm{prim}}(X)$ is a positive-definite lattice. □

6. Dense lattices

In this section, we investigate the case where $n = 4$, $l = c = 1$ and $p = r = s = 2$, and prove Theorem 1.8. We put $X := X[2,2]_1^1$ throughout this section. Note that X is of dimension 4.

The *minimal norm* $\mathrm{N}_{\min}(L)$ of a positive-definite lattice L of rank m is the minimum of norms x^2 of non-zero vectors $x \in L$, and the *normalized center density* $\delta(L)$ of L is defined by

$$\delta(L) := (\operatorname{disc} L)^{-1/2} \cdot \bigl(\mathrm{N}_{\min}(L)/4\bigr)^{m/2},$$

where $\operatorname{disc} L$ is the discriminant of L. It is known that, for each m, there exists a lattice L such that $\delta(L)$ exceeds the Minkowski–Hlawka bound

$$\zeta(m) \cdot 2^{-m+1} \cdot V_m^{-1},$$

where ζ is the Riemann zeta function and V_m is the volume of the m-dimensional unit ball. (See [3, Chap. VI] or [4, Chap. 1] for the Minkowski–Hlawka theorem.) However the proof is not constructive.

We recall the notion of *dual lattices*. Let L be a lattice. Then $L \otimes \mathbb{Q}$ is equipped with the $\mathbb{Q}$-valued symmetric bilinear form that extends the $\mathbb{Z}$-valued symmetric bilinear form on L. We define the *dual lattice* $L^{\vee}$ of L by

$$L^{\vee} := \bigl\{x \in L \otimes \mathbb{Q} \bigm| (x, y) \in \mathbb{Z} \text{ for any } y \in L\bigr\}.$$

Then $L^{\vee}$ is a $\mathbb{Z}$-module containing L as a submodule of finite index. By definition, if L_1 and L_2 are sublattices of a lattice L_3 such that $L_1 \subset L_2 \otimes \mathbb{Q}$, then L_1 is contained in $L_2^{\vee}$.

We use the notation of the previous section adapted to the present situation $n = 4$ and $p = r = s = 2$. Note that $\mathcal{M}$ is a lattice of rank $f(4) = |\mathcal{P}_0| = 85$ with the orthogonal basis $[\Sigma_{\ell(P)}]$ $(P \in \mathcal{P}_0)$. Let

$$\mathcal{N}_{\Sigma}(X) \subset \mathcal{N}_{\mathrm{prim}}(X)$$

be the submodule generated by the classes (5.5). Since $\mathcal{M}_D \subset \mathcal{N}_\Sigma(X)$, we have

$$\mathcal{M}_D \otimes \mathbb{Q} = \mathcal{N}_\Sigma(X) \otimes \mathbb{Q} = \mathcal{N}_{\mathrm{prim}}(X) \otimes \mathbb{Q}$$

by (5.6). In particular, $\mathcal{N}_\Sigma(X)$ is a lattice. Since $\mathcal{M}_D \subset \mathcal{M}$, we have $\mathcal{N}_\Sigma(X) \subset \mathcal{M} \otimes \mathbb{Q}$ in $\mathcal{N}(X) \otimes \mathbb{Q}$. We apply the above argument to $L_1 = \mathcal{N}_\Sigma(X)$, $L_2 = \mathcal{M}$, $L_3 = \mathcal{N}(X)$, and regard $\mathcal{N}_\Sigma(X)$ as embedded in the dual lattice $\mathcal{M}^\vee$.

Let e_P $(P \in \mathcal{P}_0)$ be the basis of $\mathcal{M}^\vee$ dual to the orthogonal basis $[\Sigma_{\ell(P)}]$ of $\mathcal{M}$:

$$\mathcal{M}^\vee := \bigoplus_{P \in \mathcal{P}_0} \mathbb{Z} e_P \cong \mathbb{Z}^{85}. \tag{6.1}$$

We describe the submodule $\mathcal{N}_\Sigma \subset \mathcal{M}^\vee$ in a combinatorial way using the projective geometry of $\mathcal{P}_0 = \mathbb{P}^3(\mathbb{F}_4)$. We put

$$\mathcal{P}_{k-1} := \{ S(\Lambda) \mid \Lambda \in \mathcal{L}_k \},$$

which is a subset of the power set $2^{\mathcal{P}_0}$ of $\mathcal{P}_0$. By (5.1), the vector $[\Sigma_{\ell(P)}] \in \mathcal{M} \subset \mathcal{M}^\vee$ is equal to $s^{n-2} e_P = 4e_p$, and hence we have $\mathcal{M} = s^{n-2}(\mathcal{M}^\vee) = 4\mathcal{M}^\vee$. Moreover the $\mathbb{Q}$-valued symmetric bilinear form on $\mathcal{M}^\vee$ is given by

$$(e_P, e_{P'}) = \begin{cases} 1/s^{n-2} = 1/4 & \text{if } P = P', \\ 0 & \text{if } P \neq P'. \end{cases}$$

For $S \in 2^{\mathcal{P}_0}$, we put

$$v_S := \sum_{P \in S} e_P \in \mathcal{M}^\vee.$$

By (5.1), we see that $\mathcal{N}_\Sigma$ is the submodule of $\mathcal{M}^\vee$ generated by

$$s^{3-k}(v_S - v_{S'}) \quad \bigl(S, S' \in \mathcal{P}_{k-1},\ k = 1, \dots, 3\bigr). \tag{6.2}$$

Next we introduce a code $\mathcal{C}$ over

$$R := \mathbb{Z}/s^{n-1}\mathbb{Z} = \mathbb{Z}/8\mathbb{Z}$$

and a lattice $\mathcal{M}_\mathcal{C}$. The reduction homomorphism $\mathcal{M}^\vee \to \mathcal{M}^\vee \otimes R$ is denoted by $v \mapsto \bar{v}$. Let $\mathcal{C} \subset \mathcal{M}^\vee \otimes R$ be the image of $\mathcal{N}_\Sigma \subset \mathcal{M}^\vee$ by $v \mapsto \bar{v}$. Using (6.1), we regard $\mathcal{C}$ as a submodule of $R^{f(4)} = R^{85}$, and consider $\mathcal{C}$ as an R-code of length 85. Let $\mathcal{M}_\mathcal{C} \subset \mathcal{M}^\vee$ denote the pull-back of $\mathcal{C}$ by the reduction homomorphism. Since

$$\mathcal{M}_\mathcal{C} = (\mathcal{N}_\Sigma) + 8\bigl(\mathcal{M}^\vee\bigr) = (\mathcal{N}_\Sigma) + 2(\mathcal{M}),$$

the $\mathbb{Q}$-valued symmetric bilinear form on $\mathcal{M}^\vee$ takes values in $\mathbb{Z}$ on $\mathcal{M}_\mathcal{C}$. Therefore $\mathcal{M}_\mathcal{C}$ is a lattice.

Theorem 6.1. *The lattice $\mathcal{M}_\mathcal{C}$ is an even positive-definite lattice of rank 85, with discriminant 2^{20}, and of minimal norm 8.*

Proof. Since $\mathcal{M}_{\mathcal{C}}$ is the submodule generated by the vectors (6.2) and $8e_P$ $(P \in \mathcal{P}_0)$ in $\mathcal{M}^\vee$, we can calculate the basis and the Gram matrix of $\mathcal{M}_{\mathcal{C}}$ by a computer, and confirm that $\mathcal{M}_{\mathcal{C}}$ is even and of discriminant 2^{20}. For $P \neq P'$, the vector $4(e_P - e_{P'}) \in \mathcal{M}_{\mathcal{C}}$ in (6.2) with $k = 1$ has norm 8. Hence all we have to prove is that every non-zero vector of $\mathcal{M}_{\mathcal{C}}$ is of norm $\geqslant 8$. We assume that a non-zero vector $w_s \in \mathcal{M}_{\mathcal{C}}$ satisfies ${w_s}^2 < 8$, and derive a contradiction. We express w_s as a vector in $\mathbb{Z}^{f(4)} = \mathbb{Z}^{85}$ by (6.1). Recall that $\bar{w}_s \in \mathcal{C} \subset R^{85}$ is the code word w_s mod 8.

For $\nu = 0, 1, 2, 3$, we put

$$\mathcal{K}_\nu := \operatorname{Ker}\bigl(\mathcal{C} \hookrightarrow \mathcal{M}^\vee \otimes R \to \mathcal{M}^\vee \otimes \mathbb{Z}/2^\nu\mathbb{Z}\bigr),$$

where $\mathcal{M}^\vee \otimes R \to \mathcal{M}^\vee \otimes \mathbb{Z}/2^\nu\mathbb{Z}$ is the reduction homomorphism. Then we have a filtration

$$\mathcal{C} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 = 0,$$

and each quotient

$$\Gamma_\nu := \mathcal{K}_\nu / \mathcal{K}_{\nu+1}$$

is naturally regarded as an $\mathbb{F}_2$-code of length 85.

We fix terminologies. Let $\Gamma \subset \mathbb{F}_2^N$ be an $\mathbb{F}_2$-code. The *Hamming weight* $\operatorname{wt}(\omega)$ of a code word $\omega \in \Gamma$ is the number of 1 that occurs in the components of ω. The *weight enumerator* of Γ is the polynomial $\sum_{\omega \in \Gamma} x^{\operatorname{wt}(\omega)}$.

We compute the weight enumerator of the $\mathbb{F}_2$-code

$$\Gamma_0 = \mathcal{M}_{\mathcal{C}} / \bigl(\mathcal{M}_{\mathcal{C}} \cap 2\mathcal{M}^\vee\bigr) \subset \mathcal{M}^\vee / 2\mathcal{M}^\vee = \mathcal{M}^\vee \otimes \mathbb{F}_2$$

of dimension 16 by a computer. The result is

$$1 + 3570x^{32} + 38080x^{40} + 23800x^{48} + 85x^{64}.$$

If the image of $\bar{w}_s \in \mathcal{C}$ by the projection $\mathcal{C} \to \Gamma_0$ were non-zero, then w_s would have at least 32 odd components and hence ${w_s}^2 \geqslant 8$. Thus we have $\bar{w}_s \in \mathcal{K}_1$. The $\mathbb{F}_2$-code

$$\Gamma_1 = \bigl(\mathcal{M}_{\mathcal{C}} \cap 2\mathcal{M}^\vee\bigr) / \bigl(\mathcal{M}_{\mathcal{C}} \cap 4\mathcal{M}^\vee\bigr) \subset 2\mathcal{M}^\vee / 4\mathcal{M}^\vee \cong \mathcal{M}^\vee \otimes \mathbb{F}_2$$

is of dimension 60. The weight enumerator of Γ_1 cannot be calculated directly, because 2^{60} is too large. However, the orthogonal complement $\Gamma_1^\perp$ of Γ_1 with respect to the standard inner product on $\mathbb{F}_2^{85}$ is of dimension 25, and hence its weight enumerator is calculated in a naive method. Via the MacWilliams Theorem (see [18, Ch. 5]), we see that the weight enumerator of Γ_1 is

$$\begin{aligned} &1 + 17850x^8 + 45696x^{10} + 8020600x^{12} + 229785600x^{14} + 4668633585x^{16} + \cdots \\ &\quad + 1142400x^{74} + 23800x^{76} + 357x^{80}. \end{aligned}$$

In particular, every non-zero code word of Γ_1 is of Hamming weight $\geqslant 8$. Therefore, if the image of $\bar{w}_s \in \mathcal{K}_1$ in Γ_1 were non-zero, then w_s would have at least 8 components that are congruent to 2 modulo 4, and hence ${w_s}^2 \geqslant 8$. Thus we have $\bar{w}_s \in \mathcal{K}_2$. The $\mathbb{F}_2$-code

$$\Gamma_2 = \bigl(\mathcal{M}_{\mathcal{C}} \cap 4\mathcal{M}^\vee\bigr) / \bigl(\mathcal{M}_{\mathcal{C}} \cap 8\mathcal{M}^\vee\bigr) \subset 4\mathcal{M}^\vee / 8\mathcal{M}^\vee \cong \mathcal{M}^\vee \otimes \mathbb{F}_2$$

is of dimension 84, and is defined in $\mathcal{M}^\vee \otimes \mathbb{F}_2$ by an equation

$$x_0 + \cdots + x_{84} = 0.$$

Therefore, if the image of $\bar{w}_s \in \mathcal{K}_2$ in Γ_2 were non-zero, then w_s would have at least 2 components that are congruent to 4 modulo 8, and hence $w_s{}^2 \geqslant 8$. Thus we have $\bar{w}_s \in \mathcal{K}_3$. Hence every component of w_s is congruent to 0 modulo 8. Since w_s is non-zero, we have $w_s{}^2 \geqslant 8$, which contradicts the hypothesis. $\square$

Proof of Theorem 1.8. Since $\mathcal{N}_\Sigma(X)$ is generated by the vectors (6.2), we can calculate the Gram matrix of $\mathcal{N}_\Sigma(X)$, and show that $\operatorname{disc}\mathcal{N}_\Sigma(X) = 85 \cdot 2^{16}$. On the other hand, using (5.1), (5.2) and (5.3), we can realize $\mathcal{H}$ and $\mathcal{N}(X)$ as submodules of $(\mathcal{H}+\mathcal{M}_0)^\vee$ in terms of the dual basis of the basis (5.4) of $\mathcal{H}+\mathcal{M}_0$, and compute the Gram matrix of $\mathcal{N}_{\mathrm{prim}}(X) = \mathcal{H}^\perp$. It turns out that $\mathcal{N}_{\mathrm{prim}}(X)$ is also of discriminant $85 \cdot 2^{16}$. Hence we conclude that $\mathcal{N}_\Sigma(X) = \mathcal{N}_{\mathrm{prim}}(X)$. It is easy to see that the minimal norm of $\mathcal{N}_\Sigma(X)$ is $\leqslant 8$. Since $\mathcal{N}_\Sigma(X)$ is embedded in the lattice $\mathcal{M}_\mathcal{C}$, we see that $\mathcal{N}_{\mathrm{prim}}(X)$ is even and of minimal norm $\geqslant 8$ by Theorem 6.1. $\square$

Remark 6.2. The intersection pairing of algebraic cycles on an algebraic variety in positive characteristic has been used to construct dense lattices. For example, Elkies [7–9] and Shioda [25] constructed many lattices of high density as Mordell–Weil lattices of elliptic surfaces in positive characteristics. See also [4, p. xviii].

Remark 6.3. In [23], we have obtained a dense lattice of rank 86 from the Fermat cubic 6-fold in characteristic 2. This lattice is also closely related to $\mathcal{M}_\mathcal{C}$.

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